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A WEAKLY CHAINABLE TREE-LIKE CONTINUUM WITHOUT THE FIXED POINT PROPERTY

PIOTR MINC

ABSTRACT. An example of a fixed points free map is constructed on a tree-like, weakly chainable continuum.

1. INTRODUCTION

It is still unknown whether every nonseparating plane continuum has the fixed point property. This old question was asked by Sternbach in 1935, *The Scottish Book* [21, Problem 107], but it clearly motivated even earlier papers by W. Scherrer (1926 [30]), W. L. Ayres (1930 [1]), G. Nöbeling (1932 [26]) and K. Borsuk (1932 [7]). Since then many more partial solutions, both in positive and negative directions, have been published. In particular, O. Hamilton [14] proved that every chainable continuum has the fixed point property. (A continuum is *chainable* if it is the inverse limit of an inverse sequence of arcs.) Note that every chainable continuum can be embedded in the plane (see [5]). H. Bell [2], K. Sieklucki [31] and S. Iliadis [15] proved that every nonseparating plane continuum with no indecomposable continuum in its boundary has the fixed point property. C. Hagopian [12] proved that each nonseparating arcwise connected plane continuum does not contain an indecomposable continuum in its boundary, and therefore has the fixed point property. The author proved that every nonseparating weakly chainable plane continuum also has the fixed point property [22]. (A continuum is *weakly chainable* if it is a continuous image of a chainable continuum; see [18], [10] and [25].)

Nonseparating plane continua are cell-like. In 1935, K. Borsuk [8] constructed an example of a cell-like continuum in $\mathbf{R}^3$ without the fixed point property. Two-dimensional examples were then constructed by S. Kinoshita (contractible [16]), R. Knill [17] and R. H. Bing [6]. Bing [5, p. 653] [6] asked whether a tree-like continuum without the fixed point property could be constructed. (A continuum is *tree-like* if it is the inverse limit of a sequence of trees.) D. P. Bellamy [3] answered this question affirmatively by presenting in 1978 his spectacular example.

The Bellamy continuum and other tree-like continua without the fixed point property constructed subsequently (see [27], [28], [29], [23] and [24]) are not (at least appear not to be) weakly chainable. In 1983 Bellamy asked whether every weakly chainable tree-like continuum has the fixed point property [19, Problem 36]; see also [4]. A positive answer to this question seemed likely in view of the

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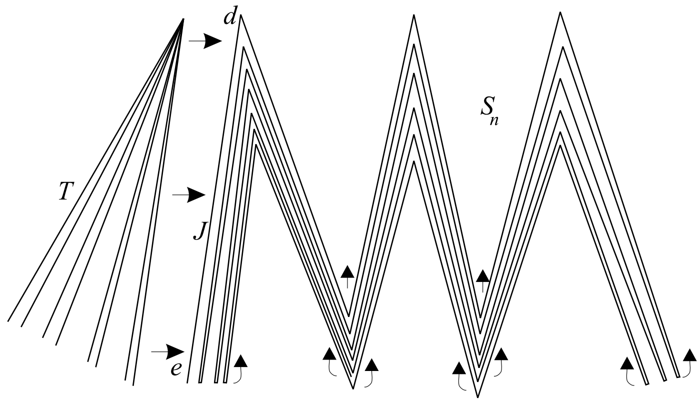

Figure 1

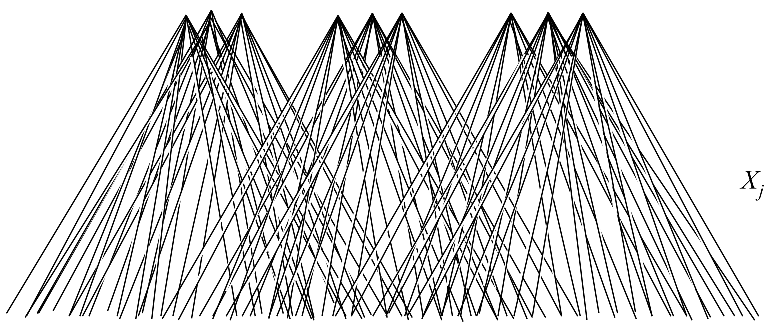

Figure 2

theorem in the plane and a classic result by K. Borsuk who, in 1954, proved the fixed point theorem for arcwise-connected tree-like continua [9]. The last result was even further extended by R. Mańka [20] to hereditarily decomposable tree-like continua. Very recently, C. Hagopian [13] proved that every map of a tree-like continuum that sends each arc-component into itself has a fixed point. It turns out, however, that Bellamy's construction can be modified to answer his question in the negative. More precisely, we prove the following theorem:

Theorem 1.1. *For each positive integer j, there is a weakly chainable tree-like continuum X_j and a map $\bar{f}_j : X_j \to X_j$ with no fixed points and no periodic points with periods less than or equal to j.*

The idea of the modification is very simple. To construct the Bellamy continuum, which is denoted here by B_j, start with S_n, a copy of the "n-fold horseshoe" contained in the xy-plane (see Figure 1). Let J denote the "basic arc" containing the endpoint e of S_n. Let d denote the other endpoint of J and let P be the plane containing J and perpendicular to the xy-plane. Remove the arc J from S_n and replace it by a cone T over a certain 0-dimensional set in such a way that T is contained in P and the vertex of T replaces d. Simultaneously, reshape the remainder of S_n by keeping the upper endpoint of each "basic arc" fixed on the xy-plane and raising the other endpoint to the level of a certain endpoint of T. In the resulting continuum B_j, each "basic arc" of $S_n \setminus J$ is almost parallel to one of the arcs forming T. To get weak chainability, we replace each "basic arc" by a copy of T (see Figure 2).

The map $\bar{f}_j$ is not a homeomorphism. Applying to $\bar{f}_j$ a technique by J. B. Fugate and L. B. Mohler [11], we get a homeomorphism of a tree-like continuum $\widetilde{B}_j$ which may not be (and probably is not) weakly chainable. For this reason, X_j and $\bar{f}_j$ cannot be used in a construction similar to the one in [24] to get a weakly chainable tree-like continuum X_j admitting a map without periodic points of all periods.

2. Construction of X_j

In this paper we will keep the notation from [23]. In particular, j, n, v, g_i, S_n, g, E, D, e, d, H_0, w, r, B_j and f_j will denote the same objects as in [23]. We will recall most of the definitions as they are needed.

Recall that v folds uniformly the real line onto the interval $[0,1]$ such that $v(i) = 0$ for each even integer i and $v(i) = 1$ for each odd i. For each positive integer k, g_k is the map stretching the interval $[0,1]$ k times and then folding it uniformly back onto itself such that $g_k(s) = v(ks)$ for each $s \in [0,1]$. Recall also that j is a positive integer and $n = 2\left(4^1-1\right)\left(4^2-1\right)\ldots\left(4^j-1\right)$.

For each $i = 0,1,\ldots$, let I_i denote the interval $[0,1]$. If $0 \le m \le i$, let $p_{mi} : I_i \to I_m$ denote the map $(g_n)^{i-m}$. Let S_n be the inverse limit of the system $\{I_i, p_{mi}\}$ and let p_i denote the projection of S_n onto I_i.

If m and i are integers such that $0 \le m \le i$, then p_{mi} restricted to the closure of any component of $(p_{mi})^{-1}((0,1))$ is a homeomorphism onto I_m. It follows that p_m restricted to the closure of any component of $(p_m)^{-1}((0,1))$ is a homeomorphism onto I_m.

For each $s \in [0,1]$, let $A_i(s)$ denote the set of $(p_{0i})^{-1}(s)$. Let A_i, E_i and D_i denote $A_i\left(\frac{1}{2}\right)$, $A_i(0)$ and $A_i(1)$, respectively.

For each $s \in I_i \setminus (E_i \cup D_i)$, let $J_i(s)$ denote the closure of the component of s in $I_i \setminus (E_i \cup D_i)$. Let $e_i(s) \in E_i$ and $d_i(s) \in D_i$ denote the endpoints of $J_i(s)$.

Observe that if m and i are integers such that $0 \le m \le i$, and $a \in A_i$, then $p_{mi}(a) \in A_m$,

$$J_m(p_{mi}(a)) = p_{mi}(J_i(a)), \tag{2.1}$$

$$e_m(p_{mi}(a)) = p_{mi}(e_i(a)) \tag{2.2}$$

and

$$d_m(p_{mi}(a)) = p_{mi}(d_i(a)). \tag{2.3}$$

Since n is even, for each positive integer i and each point $a \in A_i$, there is exactly one point $\delta_i(a) \in A_i \setminus \{a\}$ such that $d_i(a) = d_i(\delta_i(a))$. Additionally, set $d_0\left(\frac{1}{2}\right) = \frac{1}{2}$.

Observe that if m and i are integers such that $0 \le m \le i$, and $a \in A_i$, then

$$p_{mi}(\delta_i(a)) = \delta_m(p_{mi}(a)). \tag{2.4}$$

If $a \in A_i$ and $\dfrac{3}{2n^i} \le a \le 1 - \dfrac{3}{2n^i}$, then there is exactly one point $\epsilon_i(a) \in A_i \setminus \{a\}$ such that $e_i(a) = e_i(\epsilon_i(a))$. Additionally, set $\epsilon_i(a) = a$, if either $a = \dfrac{1}{2n^i}$ or $a = 1 - \dfrac{1}{2n^i}$.

Notice that if m and i are integers such that $0 \le m \le i$, and $a \in A_i$, then

$$p_{mi}(\epsilon_i(a)) = \epsilon_m(p_{mi}(a)). \tag{2.5}$$

For each $s \in [0,1]$, let $A(s)$ denote the set of $p_0^{-1}(s)$. By A, E and D we denote the sets $A\left(\frac{1}{2}\right)$, $A(0)$ and $A(1)$. Notice that E and D coincide with those defined in [23].

If $x \in S_n \setminus (E \cup D)$, let $J(x)$ denote the closure of the component of x in $S_n \setminus (E \cup D)$. If $0 \le m \le i$, then p_{mi} restricted to $J_i(p_i(x))$ is a homeomorphism onto $J_m(p_m(x))$. It follows that p_0 restricted to $J(x)$ is a homeomorphism onto $[0,1]$, and therefore $J(x)$ is an arc with one endpoint $e(x) \in E$ and the other $d(x) \in D$. Let e denote the endpoint of S_n, that is the point such that $p_i(e) = 0$ for each $i = 0, 1, \ldots$. Let a_0 denote the point of A such that $e(a_0) = e$ and let d denote $d(a_0)$. Notice that e and d coincide with those defined in [23].

Observe that, for each point $a \in A$, there is exactly one point $\delta(a) \in A \setminus \{a\}$ such that $d(a) = d(\delta(a))$. Similarly, for each point $a \in A \setminus \{a_0\}$, there is exactly one point $\epsilon(a) \in A \setminus \{a\}$ such that $e(a) = e(\epsilon(a))$. Additionally, set $\epsilon(a_0) = a_0$. Clearly, $\delta(\delta(a)) = a$ and $\epsilon(\epsilon(a)) = a$. Observe also that $\delta(a)$ and $\epsilon(a)$ depend continuously on the choice of a.

Since $p_i(A) = A_i$, $p_i(E) = E_i$, $p_i(D) = D_i$ and p_i restricted to the closure of any component of $(p_i)^{-1}((0,1))$ is a homeomorphism onto I_i, we have that

$$p_i(\delta(a)) = \delta_i(p_i(a)) \tag{2.6}$$

and

$$p_i(\epsilon(a)) = \epsilon_i(p_i(a)). \tag{2.7}$$

Let $\bar{S}_j$ be $[0,1] \times A$ with the following identifications:

$(1, a)$ is identified with $(1, \delta(a))$ for each $a \in A$ and
$(0, a)$ is identified with $(0, \epsilon(a))$ for each $a \in A \setminus \{a_0\}$.

Let $h : \bar{S}_j \to S_j$ be the map such that $h(s, a) \in J(a)$ and $p_0(h(s,a)) = s$ for each $s \in [0,1]$ and each $a \in A$. Clearly, h is a homeomorphism of $\bar{S}_j$ onto S_j.

Since $g_2 \circ g_n = g_n \circ g_2$, the maps $g_2 : I_i \to I_i$ induce a map $g : S_n \to S_n$ (see [23]). Recall that H_0, defined in [23], is a 0-dimensional compactum, $r : H_0 \to H_0$ and $w : E \setminus \{e\} \to H_0$ are continuous maps such that $r \circ w = w \circ g$.

Let T_0 be $H_0 \times [0,1]$ with $H_0 \times \{1\}$ identified to a point. We will denote by α the natural projection of $H_0 \times [0,1]$ onto T_0. Let t_0 denote the point $\alpha(H_0 \times \{1\})$. Let π_0 denote the projection of T_0 onto the interval $[0,1]$. For each point $a \in A \setminus \{a_0\}$ and each $s \in [0,1]$, let $w_s(a)$ denote the point $\alpha(w(e(a)), s)$.

Since $e(\epsilon(a)) = e(a)$, we have that $w_s(a) = w_s(\epsilon(a))$.

Let X_j be $T_0 \times A$ with the following identifications:

(t_0, a) is identified with $(t_0, \delta(a))$ for each $a \in A$ and
$(w_0(a), a)$ is identified with $(w_0(\epsilon(a)), \epsilon(a))$ for each $a \in A \setminus \{a_0\}$.

We will denote by κ the natural projection of $T_0 \times A$ onto X_j. Let $T = \kappa(T_0 \times \{a_0\})$ and $t_1 = \kappa(t_0, a_0)$.

Let $\bar{\pi} : X_j \to \bar{S}_n$ be defined in the following way: if $x = \kappa(t, a)$ for $(t, a) \in T_0 \times A$, then let $\bar{\pi}(x)$ be the point in $\bar{S}_n$ representing $(\pi_0(t), a)$. Notice that $\bar{\pi}$ is a continuous map. Let $\pi = h \circ \bar{\pi}$.

We will define $\chi : (S_n \setminus J(a_0)) \cup \{d\} \to (X_j \setminus T) \cup \{t_1\}$ by the following: if $x \in J(a)$ for $a \in A \setminus \{a_0\}$ and $s = p_0(x)$, then $\chi(x) = \kappa(w_s(a), a)$. Observe that χ is a continuous map of $(S_n \setminus J(a_0)) \cup \{d\}$ and $\pi \circ \chi$ is the identity map on $(S_n \setminus J) \cup \{d\}$. Let B_j denote $\chi((S_n \setminus J) \cup \{d\}) \cup T$. Observe that so defined B_j coincides with the one defined in [23].

3. Construction of $\bar{f}_j$

For an arbitrary point a of $A = A\left(\frac{1}{2}\right)$, let a_- and a_+ denote the points $A\left(\frac{1}{4}\right) \cap J(a)$ and $A\left(\frac{3}{4}\right) \cap J(a)$, respectively. Note that a_- and a_+ depend continuously on a. Since $g : S_n \to S_n$ is the map induced by $g_2 : I_i \to I_i$, we have that $g(a) \in D$, $g(a_-) \in A$ and $g(a_+) \in A$.

The following proposition can be easily verified and its proof is omitted here.

Proposition 3.1. *Suppose $b \in J(a)$ and $s = p_0(b)$. Then $p_0(g(b)) = g_2(s)$. Moreover, $g(b) \in J(g(a_-))$ if $0 \le s \le \frac{1}{2}$, and $g(b) \in J(g(a_+))$ if $\frac{1}{2} \le s \le 1$.*

Since $g(J(a)) = J(g(a_-)) \cup J(g(a_+))$ and $g(a) \in D$,

$$\delta(g(a_-)) = g(a_+). \tag{3.2}$$

Since $g(e(a)) \in E$ is an endpoint of $J(g(a_-))$,

$$e(g(a_-)) = g(e(a)). \tag{3.3}$$

Substituting $\epsilon(a)$ instead of a in (3.3) we get

$$e\left(g\left([\epsilon(a)]_-\right)\right) = g(e(\epsilon(a))). \tag{3.4}$$

Since $J(a) = J(a_-)$ intersects $J(\epsilon(a)) = J\left([\epsilon(a)]_-\right)$ at $e(a)$, the arcs $J(g(a_-))$ and $J\left(g\left([\epsilon(a)]_-\right)\right)$ intersect at $g(e(a)) \in E$. It follows that

$$\epsilon\left(g\left([\epsilon(a)]_-\right)\right) = g(a_-). \tag{3.5}$$

Since $J(a) = J(a_+)$ intersects $J(\delta(a)) = J\left([\delta(a)]_+\right)$ at $d(a)$, the arcs $J(g(a_+))$ and $J\left(g\left([\delta(a)]_+\right)\right)$ intersect at $g(d(a)) \in E$. It follows that

$$\epsilon\left(g\left([\delta(a)]_+\right)\right) = g(a_+). \tag{3.6}$$

We will define a function $\bar{f}_j : X_j \to X_j$ in the following way. For an arbitrary $x \in X_j$, let $a \in A$, $z \in H_0$ and $s \in [0,1]$ be such that $x = \kappa(\alpha(z,s), a)$. Let

$$\bar{f}_j(x) = \begin{cases} \kappa(\alpha(r(z), 2s), g(a_-)), & \text{for } 0 \le s \le \frac{1}{2}, \\ \kappa(w_{2-2s}(g(a_+)), g(a_+)), & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

Proposition 3.7. *Every nonendpoint of X_j is eventually moved to B_j by some iteration of $\bar{f}_j$.*

Proof of 3.7. Every nonendpoint x of X_j is either contained in B_j or it is of the form $x = \kappa(\alpha(z,s), a)$, where $s > 0$. Let k be an integer such that $s2^k \ge 1$. Observe that $\bar{f}_j^k(x) \in B_j$. □

We will show that

Proposition 3.8. *$\bar{f}_j$ is continuous.*

Proof of 3.8. Notice that the choice s is unique. Since every function appearing in the definition of $\bar{f}_j$ is continuous, it is enough to show that the function is well defined. For this purpose we need to show

3.8.1. *If $s = \frac{1}{2}$, then the first line of the definition of $\bar{f}_j$ yields the same result as the second.*

3.8.2. *The definition of $\bar{f}_j$ does not depend on the choice of a and z describing the same x.*

Proof of 3.8.1. Since $s = \frac{1}{2}$, we have that $\alpha(r(z), 2s) = \alpha(r(z), 1) = t_0$ and $w_{2-2s}(g(a_+)) = w_1(g(a_+)) = t_0$. According to the first line of the definition $\bar{f}_j(x) = \kappa(\alpha(r(z), 2s), g(a_-)) = \kappa(t_0, g(a_-)) = \kappa(t_0, \delta(g(a_-)))$. By (3.2), the last point is the same as $\kappa(t_0, g(a_+)) = \kappa(w_{2-2s}(g(a_+)), g(a_+))$ which is $\bar{f}_j(x)$ evaluated by the second line of the definition of $\bar{f}_j$. □

Proof of 3.8.2. Suppose that $a' \in A$ and $z' \in H_0$ are such that $\kappa(\alpha(z', s), a') = x = \kappa(\alpha(z, s), a)$. Let y and y' denote the values resulting from applying the definition of $\bar{f}_j$ to the triples z, s, a and z', s, a', respectively. We will show that $y' = y$. It is enough to consider the following two cases:

Case 1. $s = 0$, $a' = \epsilon(a)$ and $z = z' = w(e(a)) = w(e(\epsilon(a)))$.

Case 2. $s = 1$ and $a' = \delta(a)$.

Proof of 3.8.2 *assuming Case* 1. Observe that

$$y' = \kappa\left(\alpha(r(z'), 0), g\left([a']_-\right)\right) = \kappa\left(\alpha(r(w(e(\epsilon(a)))), 0), g\left([\epsilon(a)]_-\right)\right).$$

Since $r \circ w = w \circ g$, we have the result that

$$r(w(e(\epsilon(a)))) = w(g(e(\epsilon(a)))).$$

By (3.4),

$$w(g(e(\epsilon(a)))) = w\left(e\left(g\left([\epsilon(a)]_-\right)\right)\right)$$

and consequently $\alpha(r(w(e(\epsilon(a)))), 0) = w_0\left(g\left([\epsilon(a)]_-\right)\right)$. Thus,

$$y' = \kappa\left(w_0\left(g\left([\epsilon(a)]_-\right)\right), g\left([\epsilon(a)]_-\right)\right).$$

By the definition of X_j we get that

$$y' = \kappa\left(w_0\left(\epsilon\left(g\left([\epsilon(a)]_-\right)\right)\right), \epsilon\left(g\left([\epsilon(a)]_-\right)\right)\right).$$

By (3.5),

$$y' = \kappa(w_0(g(a_-)), g(a_-)) = \kappa(\alpha(w(e(g(a_-))), 0), g(a_-)).$$

Applying (3.3) to the last expression we get that

$$y' = \kappa(\alpha(w(g(e(a))), 0), g(a_-)).$$

Since $w \circ g = r \circ w$,

$$y' = \kappa(\alpha(r(w(e(a))), 0), g(a_-)) = y.$$

□

Proof of 3.8.2 *assuming Case* 2. It follows from the definition of X_j that

$$y' = \kappa\left(w_0\left(g\left([a']_+\right)\right), g\left([a']_+\right)\right) = \kappa\left(w_0\left(\epsilon\left(g\left([a']_+\right)\right)\right), \epsilon\left(g\left([a']_+\right)\right)\right).$$

Now, apply (3.6) to get that $\epsilon\left(g\left([a']_+\right)\right) = \epsilon\left(g\left([\delta(a)]_+\right)\right) = g(a_+)$. Thus

$$y' = \kappa\left(w_0\left(\epsilon\left(g\left([a']_+\right)\right)\right), \epsilon\left(g\left([a']_+\right)\right)\right) = \kappa(w_0(g(a_+)), g(a_+)) = y.$$

□

Remark 3.9. $\bar{f}_j$ restricted to B_j is equal to f_j defined in [23].

Proposition 3.10. *The diagram*

$$\begin{array}{ccc} X_j & \xrightarrow{\bar{f}_j} & X_j \\ \pi \big\downarrow & & \big\downarrow \pi \\ S_n & \xrightarrow{g} & S_n \end{array}$$

commutes.

Proof of 3.10. Let x be an arbitrary point in X_j. Let $a \in A$, $z \in H_0$ and $s \in [0,1]$ be such that $x = \kappa(\alpha(z,s),a)$. $\pi(x)$ is the only point of $J(a)$ such that $p_0(\pi(x)) = s$. By Proposition 3.1, $p_0(g(\pi(x))) = g_2(s)$. Recall that $g_2(s) = 2s$ if $0 \le s \le \frac{1}{2}$, and $g_2(s) = 2 - 2s$ if $\frac{1}{2} \le s \le 1$.

If $0 \le s \le \frac{1}{2}$, then $g(\pi(x)) \in J(g(a_-))$. In this case,

$$\bar{f}_j(x) = \kappa(\alpha(r(z), 2s), g(a_-)).$$

It follows that $\pi\left(\bar{f}_j(x)\right)$ also belongs to $J(g(a_-))$ and $p_0\left(\pi\left(\bar{f}_j(x)\right)\right) = 2s = g_2(s)$. So, $\pi\left(\bar{f}_j(x)\right) = g(\pi(x))$ if $0 \le s \le \frac{1}{2}$.

If $\frac{1}{2} \le s \le 1$, then $g(\pi(x)) \in J(g(a_+))$. In this case,

$$\bar{f}_j(x) = \kappa(w_{2-2s}(g(a_+)), g(a_+)) = \kappa(\alpha(w(e(g(a_+))), 2-2s), g(a_+)).$$

It follows that $\pi\left(\bar{f}_j(x)\right)$ belongs to $J(g(a_+))$ and $p_0\left(\pi\left(\bar{f}_j(x)\right)\right) = 2 - 2s = g_2(s)$. So, $\pi\left(\bar{f}_j(x)\right) = g(\pi(x))$. □

Proposition 3.11. *$\bar{f}_j$ has no fixed points and no periodic points with periods less than or equal to j. Moreover, all periodic points of $\bar{f}_j$ are contained in B_j.*

Proof of 3.11. The first part of the proposition follows from Proposition 3.10 and Propositions 2.6 and 2.10 in [23]. The second part of Proposition 3.11 is a simple consequence of 3.7, 3.9 above and 2.10 in [23]. □

4. Inverse limit description of X_j

There exists a sequence $\mathcal{K}_0, \mathcal{K}_1 \ldots$ such that for each $i = 0, 1, \ldots$ the following conditions are satisfied:

$\mathcal{K}_i$ is a collection of mutually disjoint nonempty, closed subsets of H_0 filling up H_0,
$\mathcal{K}_{i+1}$ refines $\mathcal{K}_i$ and
$\lim_{i \to \infty} \operatorname{mesh}(\mathcal{K}_i) = 0$.

For $0 \le m \le i$ and $u \in \mathcal{K}_i$, let $\varphi_{mi}(u)$ be the element of $\mathcal{K}_m$ containing u. Notice that H_0 may be naturally considered as the inverse limit of the system $\{\mathcal{K}_i, \varphi_{mi}\}$. Let φ_i be the projection of H_0 onto $\mathcal{K}_i$. Observe that if $x \in H_0$ then $\varphi_i(x)$ is the unique element of $\mathcal{K}_i$ containing x.

We will construct a strictly increasing sequence of integers $\mu(0), \mu(1), \mu(2), \ldots$ such that for each positive integer i and each $b \in E_{\mu(i)} \setminus \left(p_{\mu(i-1)\,\mu(i)}\right)^{-1}(0)$, the set $w\left(\left(p_{\mu(i)}\right)^{-1}(b)\right)$ is contained in a single element $w^i(b)$ of $\mathcal{K}_{\mu(i-1)}$.

Let $\mu(0) = 1$. Suppose $\mu(i-1) = m$ has been constructed. We will construct $\mu(i)$. Since the set $E \setminus (p_m)^{-1}(0)$ is compact, there is a positive number η such that if a set $S \subset E \setminus (p_m)^{-1}(0)$ has the diameter less than η then $w(S)$ is contained in a single element of $\mathcal{K}_m$. There is an integer $k > m$ such that diam $\left((p_k)^{-1}(b)\right) < \eta$ for each $b \in I_k$. If $b \in E_k \setminus (p_{mk})^{-1}(0)$, then $(p_k)^{-1}(b) \subset E \setminus (p_m)^{-1}(0)$, and consequently $w\left((p_k)^{-1}(b)\right)$ is contained in a single element of $\mathcal{K}_m$. So, we may set $\mu(i) = k$.

In order to simplify the notation we will write p^*_{mi}, p^*_i, ϵ^*_i, δ^*_i and e^*_i instead of $p_{\mu(m)\,\mu(i)}$, $p_{\mu(i)}$, $\epsilon_{\mu(i)}$, $\delta_{\mu(i)}$ and $e_{\mu(i)}$, respectively.

Let G_i denote the set $\left\{a \in A_{\mu(i)} \mid e^*_i(a) \in \left(p^*_{i-1\,i}\right)^{-1}(0)\right\}$. Since

$$e^*_i(a) = e^*_i(\epsilon^*_i(a)),$$

we have that

Proposition 4.1. *$a \in G_i$ if and only if $\epsilon^*_i(a) \in G_i$.*

Proposition 4.2. *Suppose m and i are integers such that $1 \le m \le i$. Suppose also a is an element of $A_{\mu(i)} \setminus G_i$ such that $p^*_{mi}(a) \notin G_m$. Then*

$$\varphi_{\mu(m-1)\,\mu(i-1)}\left(w^i(e^*_i(a))\right) = w^m(e^*_m(p^*_{mi}(a))).$$

Proof of 4.2. Since $p^*_m = p^*_{mi} \circ p^*_i$, we get from (2.2) that

$$(p^*_i)^{-1}(e^*_i(a)) \subset (p^*_m)^{-1}(e^*_m(p^*_{mi}(a))).$$

Thus $w^m(e^*_m(p^*_{mi}(a)))$, the element of $\mathcal{K}_{\mu(m-1)}$ containing $(p^*_m)^{-1}(e^*_m(p^*_{mi}(a)))$, must contain $w^i(e^*_i(a))$, the element of $\mathcal{K}_{\mu(i-1)}$ containing $(p^*_i)^{-1}(e^*_i(a))$. □

Proposition 4.3. *Suppose $a \in A$ and $i = 0, 1, \ldots$ are such that $p^*_i(a) \notin G_i$. Then*

$$\varphi_{\mu(i-1)}(w(e(a))) = w^i(e^*_i(p^*_i(a))).$$

Proof of 4.3. Since $p^*_i(e(a)) = e^*_i(p^*_i(a))$, we have that

$$e(a) \in (p^*_i)^{-1}(e^*_i(p^*_i(a))).$$

Thus $w^i(e^*_i(p^*_i(a)))$ is the element of $\mathcal{K}_{\mu(i-1)}$ containing $w(e(a))$. □

For each positive integer i, let T^i_0 be $\mathcal{K}_{\mu(i-1)} \times [0,1]$ with $\mathcal{K}_{\mu(i-1)} \times \{1\}$ identified to a point. We will denote by α_i the natural projection of $\mathcal{K}_{\mu(i-1)} \times [0,1]$ onto T^i_0. Let t^i_0 denote the point $\alpha_i\left(\mathcal{K}_{\mu(i-1)} \times \{1\}\right)$.

For $1 \le m \le i$ and $z_i \in T^i_0$ we will define $\psi_{mi}(z_i) \in T^m_0$ by choosing $u_i \in \mathcal{K}_{\mu(i-1)}$ and $s \in [0,1]$ such that $\alpha_i(u_i, s) = z_i$ and setting

$$\psi_{mi}(z_i) = \alpha_m\left(\varphi_{\mu(m-1)\,\mu(i-1)}(u_i), s\right).$$

Note that ψ_{mi} is a well-defined map of T^i_0 onto T^m_0. Observe also that T_0 may be naturally considered as the inverse limit of the system $\left\{\mathcal{T}^i_0, \psi_{mi}\right\}$. We will denote by ψ_i the projection of T_0 onto T^i_0.

Suppose $a \in A_{\mu(i)} \setminus G_i$. Clearly, $e^*_i(a) \in E_{\mu(i)} \setminus \left(p^*_{i-1\,i}\right)^{-1}(0)$ and $w^i(e^*_i(a))$ is defined. Let $w^i_0(a)$ denote $\alpha_i\left(w^i(e^*_i(a)), 0\right)$. Since $e^*_i(a) = e^*_i(\epsilon^*_i(a))$, it is true that $w^i_0(a) = w^i_0(\epsilon^*_i(a))$.

Proposition 4.4. *Suppose m and i are integers such that $1 \le m \le i$. Let $a \in A_{\mu(i)}$ be such that $p^*_{mi}(a) \notin G_m$. Then*

$$\psi_{mi}\left(w^i_0(a)\right) = w^m_0(p^*_{mi}(a)).$$

Proof of 4.4. Since $p^*_{mi}(a) \notin G_m$, it follows that $a \notin G_i$ and consequently both $w^i_0(a)$ and $w^m_0(p^*_{mi}(a))$ are well-defined. Observe that

$$\psi_{mi}\left(w^i_0(a)\right) = \psi_{mi}\left(\alpha_i\left(w^i\left(e^*_i(a)\right), 0\right)\right) = \alpha_m\left(\varphi_{\mu(m-1)\,\mu(i-1)}\left(w^i\left(e^*_i(a)\right)\right), 0\right).$$

By 4.2, the last expression is equal to $\alpha_m\left(w^m\left(e^*_m\left(p^*_{mi}(a)\right)\right), 0\right) = w^m_0\left(p^*_{mi}(a)\right)$. □

Proposition 4.5. *Suppose $a \in A$ and i is a positive integer such that $p^*_i(a) \notin G_i$. Then*

$$\psi_i\left(w_0(a)\right) = w^i_0\left(p^*_i(a)\right).$$

Proof of 4.5. $\psi_i\left(w_0(a)\right) = \psi_i\left(\alpha\left(w\left(e(a)\right), 0\right)\right) = \alpha_i\left(\varphi_{\mu(i-1)}\left(w\left(e(a)\right)\right), 0\right)$. By 4.3, the last expression is equal to

$$\alpha_i\left(w^i\left(e^*_i\left(p^*_i(a)\right)\right), 0\right) = w^i_0\left(p^*_i(a)\right).$$

□

For each positive integer i, let X^i_j be $T^i_0 \times A_{\mu(i)}$ with the following identifications:

$\left(t^i_0, a\right)$ is identified with $\left(t^i_0, \delta^*_i(a)\right)$ for each $a \in A_{\mu(i)}$,
$\left(w^i_0(a), a\right)$ is identified with $\left(w^i_0\left(\epsilon^*_i(a)\right), \epsilon^*_i(a)\right)$ for each $a \in A_{\mu(i)} \setminus G_i$ and
(u, a) is identified with $\left(u, \epsilon^*_i(a)\right)$ for each $u \in T^i_0$ and each $a \in G_i$.

We will denote by κ_i the natural projection of $T^i_0 \times A_{\mu(i)}$ onto X^i_j.

The following proposition follows from (2.4).

Proposition 4.6. *Suppose m and i are integers such that $1 \leq m \leq i$. Let $a \in A_i$. Then*

$$\kappa_m\left(\psi_{mi}\left(t^i_0\right), p^*_{mi}(a)\right) = \kappa_m\left(\psi_{mi}\left(t^i_0\right), p^*_{mi}\left(\delta^*_m(a)\right)\right).$$

Proposition 4.7. *Suppose m and i are integers such that $1 \leq m \leq i$. Let $a \in A_{\mu(i)} \setminus G_i$. Then*

$$\kappa_m\left(\psi_{mi}\left(w^i_0(a)\right), p^*_{mi}(a)\right) = \kappa_m\left(\psi_{mi}\left(w^i_0\left(\epsilon^*_i(a)\right)\right), p^*_{mi}\left(\epsilon^*_i(a)\right)\right).$$

Proof of 4.7. Suppose $p^*_{mi}(a) \in G_m$. Then the point $\psi_{mi}\left(w^i_0(a), p^*_{mi}(a)\right)$ is identified with $\psi_{mi}\left(w^i_0(a), \epsilon^*_m\left(p^*_{mi}(a)\right)\right)$ by κ_m. By (2.5), $p^*_{mi}\left(\epsilon^*_i(a)\right) = \epsilon^*_m\left(p^*_{mi}(a)\right)$. Since $w^i_0\left(\epsilon^*_i(a)\right) = w^i_0(a)$, the proposition is in this case trivial. Thus, we may assume that $p^*_{mi}(a) \notin G_m$.

By Proposition 4.4,

$$\kappa_m\left(\psi_{mi}\left(w^i_0(a)\right), p^*_{mi}(a)\right) = \kappa_m\left(w^m_0\left(p^*_{mi}(a)\right), p^*_{mi}(a)\right).$$

Applying Proposition 4.1 and (2.5) we get that $p^*_{mi}\left(\epsilon^*_i(a)\right) \notin G_m$. Again by Proposition 4.4,

$$\kappa_m\left(\psi_{mi}\left(w^i_0\left(\epsilon^*_i(a)\right)\right), p^*_{mi}\left(\epsilon^*_i(a)\right)\right) = \kappa_m\left(w^m_0\left(p^*_{mi}\left(\epsilon^*_i(a)\right)\right), p^*_{mi}\left(\epsilon^*_i(a)\right)\right).$$

Applying 2.5, we get that the last expression is equal to

$$\kappa_m\left(w^m_0\left(\epsilon^*_m\left(p^*_{mi}(a)\right)\right), \epsilon^*_m\left(p^*_{mi}(a)\right)\right).$$

Since $\left(w^m_0\left(p^*_{mi}(a)\right), p^*_{mi}(a)\right)$ and $\left(w^m_0\left(\epsilon^*_m\left(p^*_{mi}(a)\right)\right), \epsilon^*_m\left(p^*_{mi}(a)\right)\right)$ are identified by κ_m, the proposition is true. □

For any two integers m and i such that $1 \leq m \leq i$, we will define $\sigma_{mi} : X_j^i \to X_j^m$ in the following way. For an arbitrary $x \in X_j^i$, let $a \in A_{\mu(i)}$ and $z \in T_0^i$ be such that $x = \kappa_i(z, a)$. Let $\sigma_{mi}(x) = \kappa_m(\psi_{mi}(z), p_{mi}^*(a))$. It follows from 4.6 and 4.7 that σ_{mi} is a well-defined continuous map.

Proposition 4.8. *The continuum X_j is homeomorphic to the inverse limit of the system $\{X_j^i, \sigma_{mi}\}$.*

Proof of Proposition 4.8. For each positive integer i, we will define $\sigma_i : X_j \to X_j^i$ in the following way. For an arbitrary $x \in X_j$, take $z \in T_0$ and $a \in A$ such that $x = \kappa(z, a)$. Let $\sigma_i(x) = \kappa_i(\psi_i(z), p_i^*(a))$. An easy proof that σ_i is a well-defined map, not depending on the choice of z and a, follows from (2.6), (2.7) and Proposition 4.3, and will be omitted here.

Claim. Suppose $x = \kappa(z, a)$ and $x' = \kappa(z', a')$ are two different points of X_j. Then there is a positive integer k such that $\sigma_k(x) \neq \sigma_k(x')$.

Proof of Claim. Since A and T_0 are the inverse limits of the systems $\{A_{\mu(i)}, p_{mi}^*\}$ and $\{T_0^i, \psi_{mi}\}$, respectively, there is a positive integer k satisfying the following conditions:

1. if $a' \neq a$, then $p_k^*(a') \neq p_k^*(a)$,
2. if $a' \neq \delta(a)$, then $p_k^*(a') \neq p_k^*(\delta(a))$,
3. if $a' \neq \epsilon(a)$, then $p_k^*(a') \neq p_k^*(\epsilon(a))$,
4. if $a \neq a_0$, then $p_k^*(a) \notin G_k$,
5. if $a' \neq a_0$, then $p_k^*(a') \notin G_k$,
6. if $z' \neq z$, then $\psi_k(z') \neq \psi_k(z)$.

Now, it is easy to prove using (2.6) and (2.7) that the points $(\psi_k(z), p_k^*(a))$ and $(\psi_k(z'), p_k^*(a'))$ are not among the pairs of points identified by κ_k. □

Observe that for each integer m and i such that $1 \leq m \leq i$, we have that $\sigma_m = \sigma_{mi} \circ \sigma_i$. For each $x \in X_j$, let $\sigma(x) = (\sigma_1(x), \sigma_2(x), \dots)$. Clearly, σ is a continuous map of X_j into the inverse limit of the system $\{X_j^i, \sigma_{mi}\}$. Since σ_i maps X_j onto X_j^i, the map σ is surjective. It follows from the claim that σ is a homeomorphism. □

Let $\mathcal{C}_i$ denote the collection of sets $\kappa_i(T_0^i \times \{a\})$ where $a \in A_{\mu(i)}$. Clearly, if $\kappa_i(T_0^i \times \{a\}) = \kappa_i(T_0^i \times \{a'\})$, then both a and a' belong to G_i and $a' = \epsilon_i^*(a)$. Let $\lambda(i)$ be the number of distinct elements of $\mathcal{C}_i$. Arrange elements of $\mathcal{C}_i$ into a sequence $C_i^1, C_i^2, \dots, C_i^{\lambda(i)}$ in such a way that if k and k' are integers and a, a' are two elements of $A_{\mu(i)}$ such that $1 \leq k < k' \leq \lambda(i)$, $C_i^k = \kappa_i(T_0^i \times \{a\})$ and $C_i^{k'} = \kappa_i(T_0^i \times \{a'\})$, then $a < a'$.

Observe that for each integer $i \geq 2$, $\sigma_{i-1\,i}$ maps each element of $\mathcal{C}_i$ onto an element of $\mathcal{C}_{i-1}$. For $k = 1, \dots, \lambda(i)$, let $\tau(k, i)$ denote the integer such that $1 \leq \tau(k, i) \leq \lambda(i-1)$ and $\sigma_{i-1\,i}(C_i^k) = C_{i-1}^{\tau(k,i)}$. Clearly, $|\tau(k, i) - \tau(k-1, i)| \leq 1$ for each $k = 2, \dots, \lambda(i)$. Observe also that, for each $k = 1, \dots, \lambda(i) - 1$, the intersection $C_i^k \cap C_i^{k+1}$ consists of one point. We will denote this point by c_i^k.

The following proposition is a simple consequence of the construction.

Proposition 4.9. *Suppose L' and L are closed intervals contained in the real line. Let b_0 and b_1 be the endpoints of L. Suppose β' is a map of L' onto $C_{i-1}^{\tau(k,i)}$. Let $c_0, c_1 \in C_i^k$ and $b_0', b_1' \in L'$ be such that $\sigma_{i-1\,i}(c_0) = \beta'(b_0')$ and $\sigma_{i-1\,i}(c_1) = \beta'(b_1')$*

Then there is a map $\gamma_k : L \to L'$ and there is a map β_k of L onto C_i^k such that $\gamma_k(b_0) = b_0'$, $\gamma_k(b_1) = b_1'$, $\beta_k(b_0) = c_0$, $\beta_k(b_1) = c_1$ and $\beta' \circ \gamma_k = \sigma_{i-1\,i} \circ \beta_k$.

Proof of 4.9. Let a_i^k denote an element of $A_{\mu(i)}$ such that $C_i^k = \kappa_i\left(T_0^i \times \left\{a_i^k\right\}\right)$. Let $\mathcal{Z}$ be the collection of components of $C_i^k \setminus \left\{\kappa_i\left(t_0^i, a_i^k\right)\right\}$. Let $\{L_z\}_{z\in\mathcal{Z}}$ be a collection of mutually disjoint subintervals contained in the interior of L. Let $\gamma_k : L \to L'$ be such that $\gamma_k(b_0) = b_0'$, $\gamma_k(b_1) = b_1'$ and $\gamma_k(L_z) = L'$ for each $z \in \mathcal{Z}$.

Since $\sigma_{i-1\,i}$ restricted to each $z \in \mathcal{Z}$ is a homeomorphism onto $\sigma_{i-1\,i}(z)$, the map $\beta' \circ \gamma_k$ can be lifted through $\sigma_{i-1\,i}$ to map $\beta_k : L \to C_i^k$ such that $\beta_k(b_0) = c_0$, $\beta_k(b_1) = c_1$ and $\beta' \circ \gamma_k = \sigma_{i-1\,i} \circ \beta_k$. The map β_k can be made surjective by guaranteeing that $z \subset \beta_k(L_z)$ for each $z \in \mathcal{Z}$. □

Proposition 4.10. *Let s_m denote $\dfrac{m}{\lambda(i-1)}$ for $m = 0, 1, \ldots, \lambda(i-1)$ and let u_k denote $\dfrac{k}{\lambda(i)}$ for $k = 0, 1, \ldots, \lambda(i)$. Suppose ω_{i-1} is a map of the interval $[0,1]$ onto X_j^{i-1} such that $\omega_{i-1}([s_{m-1}, s_m]) = C_{i-1}^m$ for $m = 1, \ldots, \lambda(i-1)$. Then there are maps $\omega_i : [0,1] \to X_j^i$ and $\xi_i : [0,1] \to [0,1]$ such that $\sigma_{i-1\,i} \circ \omega_i = \omega_{i-1} \circ \xi_i$ and $\omega_i([u_{k-1}, u_k]) = C_i^k$ for $k = 1, \ldots, \lambda(i)$.*

Proof of 4.10. Observe that $\omega_{i-1}(s_m) = c_{i-1}^m$ for $m = 1, \ldots, \lambda(i-1) - 1$.

For each $k = 1, \ldots, \lambda(i)$, we will define two maps:

a map $\gamma_k : [u_{k-1}, u_k] \to \left[s_{\tau(k,i)-1}, s_{\tau(k,i)}\right]$ and
a map β_k of $[u_{k-1}, u_k]$ onto C_i^k such that the diagram

(4.10.1)
$$\begin{array}{ccc} \left[s_{\tau(k,i)-1}, s_{\tau(k,i)}\right] & \xleftarrow{\gamma_k} & [u_{k-1}, u_k] \\ \omega_{i-1}\downarrow & & \downarrow\beta_k \\ C_{i-1}^{\tau(k,i)} & \xleftarrow{\sigma_{i-1\,i}} & C_i^k \end{array}$$

is commutative,

(4.10.2) $\beta_k(u_k) = c_i^k$ for $k = 1, \ldots, \lambda(i) - 1$,
(4.10.3) $\beta_k(u_{k-1}) = c_i^{k-1}$ for $k = 2, \ldots, \lambda(i)$,
(4.10.4) $\gamma_k(u_{k-1}) = \gamma_{k-1}(u_{k-1})$ for $k = 2, \ldots, \lambda(i)$,
(4.10.5) if $\tau(k,i) < \tau(k+1,i)$, then $\gamma_k(u_k) = s_{\tau(k,i)}$ for $k = 1, \ldots, \lambda(i) - 1$ and
(4.10.6) if $\tau(k,i) > \tau(k+1,i)$, then $\gamma_k(u_k) = s_{\tau(k,i)-1}$ for $k = 1, \ldots, \lambda(i) - 1$.

In order to construct β_1 and γ_1, set $k = 1$, $L' = [s_0, s_1]$, $b_0 = u_0$, $b_1 = u_1$, $\beta' = \omega_{i-1} \restriction L'$, $b_0' = s_0$, $b_1' = s_1$ and $c_1 = c_i^1$. Additionally, let c_0 be a point of C_i^1 such that $\sigma_{i-1\,i}(c_0) = \omega_{i-1}(s_0)$. Now, apply Proposition 4.9 to get β_1 and γ_1.

Suppose $\beta_1, \ldots, \beta_{k-1}$ and $\gamma_1, \ldots, \gamma_{k-1}$ satisfying (4.10.1)-(4.10.6) have been constructed. We will observe that

(4.10.7)
$$\gamma_{k-1}(u_{k-1}) \in \left[s_{\tau(k,i)-1}, s_{\tau(k,i)}\right].$$

One of the following three cases is true: either

1 $\tau(k-1,i) = \tau(k,i) - 1$ or
2 $\tau(k-1,i) = \tau(k,i)$ or
3 $\tau(k-1,i) = \tau(k,i) + 1$.

In the first case, by (4.10.5) for $k-1$, $\gamma_{k-1}(u_{k-1}) = s_{\tau(k-1,i)} = s_{\tau(k,i)-1}$. (4.10.7) is obvious in the second case. Finally, in the third case, by (4.10.6) for $k-1$, $\gamma_{k-1}(u_{k-1}) = s_{\tau(k-1,i)-1} = s_{\tau(k,i)}$.

We will now construct β_k and γ_k by applying Proposition 4.9 again. Set $L' = \left[s_{\tau(k,i)-1}, s_{\tau(k,i)}\right]$, $b_0 = u_{k-1}$, $b_1 = u_k$ and $\beta' = \omega_{i-1} \restriction L'$. Let $b'_0 = \gamma_{k-1}(u_{k-1})$ and $c_0 = c_i^{k-1}$. By (4.10.7), $b'_0 \in L'$. It follows from (4.10.1) and (4.10.2) for $k-1$ that $\sigma_{i-1\,i}(c_0) = \beta'(b'_0)$.

Let $c_1 = c_i^k$ if $k < \lambda(i)$ and let c_1 be any point of C_i^k if $k = \lambda(i)$. Before we apply Proposition 4.9 we must define b'_1 in such a way that $\sigma_{i-1\,i}(c_1) = \beta'(b'_1)$. We will consider the following three cases:

Case 1: either $k = \lambda(i)$ or $\tau(k,i) = \tau(k+1,i)$,
Case 2: $\tau(k,i) = \tau(k+1,i) - 1$ and
Case 3: $\tau(k,i) = \tau(k+1,i) + 1$.

In the first case b'_1 may be defined as any element of L' such that $\omega_{i-1}(b'_1) = \sigma_{i-1\,i}(c_1)$. In the second case, set $b'_1 = s_{\tau(k,i)}$ and notice that $\omega_{i-1}(b'_1) = c_{i-1}^{\tau(k,i)} = \sigma_{i-1\,i}\left(c_i^k\right)$. Finally, in the third case, set $b'_1 = s_{\tau(k,i)-1}$ and notice that $\omega_{i-1}(b'_1) = c_{i-1}^{\tau(k,i)-1} = \sigma_{i-1\,i}\left(c_i^k\right)$.

Now, observe that Proposition 4.9 yields β_k and γ_k satisfying (4.10.1)-(4.10.6).

Let the functions ω_i and ξ_i be defined by $\omega_i(z) = \beta_k(z)$ and $\xi_i(z) = \gamma_k(z)$ for $z \in [u_{k-1}, u_k]$, where $k = 1, \ldots, \lambda(i)$. It follows from (4.10.1)-(4.10.4) that ω_i and ξ_i are well-defined maps satisfying the proposition. □

Proposition 4.11. *X_j is weakly chainable.*

Proof of 4.11. Let ω_0 be a map of $[0,1]$ onto X_j^0. Use Proposition 4.10 repeatedly to get two sequences of maps $\xi_1, \xi_2, \ldots$ and $\omega_1, \omega_2, \ldots$ such that $\xi_i : [0,1] \to [0,1]$, ω_i maps $[0,1]$ onto X_j^i for each positive integer i and the diagram

$$\begin{array}{ccccccc} [0,1] & \xleftarrow{\xi_1} & [0,1] & \xleftarrow{\xi_2} & [0,1] & \xleftarrow{\xi_3} & \ldots \\ \downarrow{\scriptstyle \omega_0} & & \downarrow{\scriptstyle \omega_1} & & \downarrow{\scriptstyle \omega_2} & & \downarrow \\ X_j^0 & \xleftarrow{\sigma_{01}} & X_j^1 & \xleftarrow{\sigma_{12}} & X_j^2 & \xleftarrow{\sigma_{23}} & \ldots \end{array}$$

is commutative. The diagram induces a map from the inverse limit of the sequence $\{[0,1], \xi_i\}$ onto X_j which is the inverse limit of $\left\{X_j^i, \sigma_{mi}\right\}$. □

The proof of Theorem 1.1 follows from 3.8, 3.11, 4.8 and 4.11.

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Department of Mathematics, Auburn University, Auburn, Alabama 36849
E-mail address: `mincpio@mail.auburn.edu`